# SOLUTIONS MANUAL

## CHAPTER 1

**1** There are many ways to do this. One possibility for **A** is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{5}{2} & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 1 & -\frac{3}{2} & -8 & 11 \\ 0 & 1 & 5 & -7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Delta^{-} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \mathbf{Q}\Delta^{-}\mathbf{P} = \begin{bmatrix} 8 & -3 & 0 \\ -5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

One possibility for **B** is

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{bmatrix}, \quad \mathbf{Q} = \begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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$$\Delta^- = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G} = \mathbf{Q}\Delta^-\mathbf{P} = \begin{bmatrix} -\frac{19}{6} & \frac{11}{3} & -\frac{3}{2} & 0 \\ \frac{7}{3} & -\frac{7}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} & 0 \end{bmatrix}$$

**2** There are as many generalized inverses to be found by this method as there are non-singular minors of order the rank of the matrix.

One possibility for **A** is to use the $2 \times 2$ minor in the upper right-hand corner. Its inverse is $\begin{bmatrix} 8 & -3 \\ -5 & 2 \end{bmatrix}$. The resulting generalized inverse is $\mathbf{G} = \begin{bmatrix} 8 & -3 & 0 \\ -5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

One possibility for **B** is to use the minor $\mathbf{M} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$. Its inverse is $\begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 \\ -\frac{2}{3} & \frac{11}{3} & -2 \\ 1 & -2 & 1 \end{bmatrix}$. The resulting generalized inverse is $\mathbf{G} = \begin{bmatrix} -\frac{2}{3} & -\frac{4}{3} & 1 & 0 \\ -\frac{2}{3} & \frac{11}{3} & -2 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

**3** **(a)** The general solution takes the form $\mathbf{x} = \mathbf{G}\mathbf{y} + (\mathbf{G}\mathbf{A} - \mathbf{I})\mathbf{z}$. Using the generalized inverse in of A 2, we have

$$\tilde{\mathbf{x}} = \begin{bmatrix} 8 & -3 & 0 \\ -5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -13 \\ -11 \end{bmatrix}$$

$$+ \left( \begin{bmatrix} 8 & -3 & 0 \\ -5 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & -1 \\ 5 & 8 & 0 & 1 \\ 1 & 2 & -2 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix}$$

$$= \begin{bmatrix} 31 + 8z_3 - 11z_4 \\ -21 - 5z_3 + 7z_4 \\ -z_3 \\ -z_4 \end{bmatrix}$$

**(b)** Using the generalized inverse of **B** in 1 we get in a similar manner

$$\tilde{\mathbf{x}} = \begin{bmatrix} -8 - 5z_4 \\ 11 - 6z_4 \\ 2z_4 \\ -z_4 \end{bmatrix}$$

**4** We have that

$$\mathbf{A}'\mathbf{A} = \begin{bmatrix} 6 & 1 & 11 \\ 1 & 11 & -9 \\ 11 & -9 & 31 \end{bmatrix}$$

By the Cayley–Hamilton theorem,

$$390(\mathbf{A}'\mathbf{A}) - 48(\mathbf{A}'\mathbf{A})^2 + (\mathbf{A}'\mathbf{A})^3 = 0$$

Then

$$\mathbf{T} = (-1/390)(-48\mathbf{I} + (\mathbf{A}'\mathbf{A})) = \begin{bmatrix} \frac{7}{65} & -\frac{1}{390} & -\frac{11}{390} \\ -\frac{1}{390} & \frac{37}{390} & \frac{3}{130} \\ -\frac{11}{390} & \frac{3}{130} & \frac{17}{390} \end{bmatrix}$$

Then the Moore–Penrose inverse is

$$\mathbf{K} = \mathbf{TA}' = \begin{bmatrix} \frac{2}{39} & \frac{1}{13} & \frac{1}{39} & \frac{5}{39} \\ \frac{17}{390} & \frac{1}{65} & \frac{14}{195} & \frac{101}{390} \\ \frac{23}{390} & \frac{9}{65} & -\frac{4}{195} & -\frac{1}{390} \end{bmatrix}$$

**5** By direct computation, we see that only Penrose condition (ii) is satisfied.

**6** **(a)** For $\mathbf{M_1}$,

$$\mathbf{G}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{3}{2} & \frac{5}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

For $\mathbf{M}_2$,

$$\mathbf{G}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \\ 0 & \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

For $\mathbf{M}_3$,

$$\mathbf{G}_3 = \begin{bmatrix} \frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{3}{20} & 0 & \frac{1}{5} \end{bmatrix}$$

**(b)** A generalized inverse is

$$\mathbf{G} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 11 & -4 \\ 0 & -4 & \frac{3}{2} \end{bmatrix}.$$

There are infinitely many other correct answers.

**7** **(a)** $\mathbf{A}'\mathbf{A} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}, \mathbf{AA}' = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

Non-zero eigenvalue = 4 for both matrices. Eigenvectors

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \text{Thus, } \mathbf{U}' = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{S} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \mathbf{\Lambda} = \begin{bmatrix} 4 \end{bmatrix}$$

$$\mathbf{A}^+ = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

Alternatively, by the Cayley–Hamilton Theorem

$$\mathbf{T} = \frac{1}{4}\mathbf{I}, \mathbf{K} = \mathbf{TA}' = \begin{bmatrix} -\frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

**(b)** By direct matrix multiplication, we find that (i) satisfies conditions (i) and (ii) so it is a reflexive generalized inverse, (ii) satisfies conditions (i) and (iv) so it is a least square generalized inverse, (iii) satisfies conditions (i) and (iii) so it is a minimum norm geneeralized inverse and (iv) satisfies conditions (i), (iii), and (iv) so it is both a least-square and minimum norm generalized inverse but not reflexive.

**8** There are a number of right answers to part (a) and (b) depending on the choice of generalized inverse.

**(a)** We have that

$$\mathbf{XX}' = \begin{bmatrix} 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 \end{bmatrix}, (\mathbf{XX}')^- = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{X}_{mn} = \mathbf{X}'(\mathbf{XX}')^- = \begin{bmatrix} 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{bmatrix}.$$

**(b)** We have that $\mathbf{X}'\mathbf{X} = \begin{bmatrix} 6 & 3 & 3 \\ 3 & 3 & 0 \\ 3 & 0 & 3 \end{bmatrix}, (\mathbf{X}'\mathbf{X})^- = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$ and $\mathbf{X}_{ls} =$

$$(\mathbf{X}'\mathbf{X})^-\mathbf{X}' = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

**(c)** Both the minimum norm and least-square inverses are reflexive. We have

$$\mathbf{X}^+ = \mathbf{X}_{mn}\mathbf{XX}_{ls} = \begin{bmatrix} \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} & \frac{1}{9} \\ \frac{2}{9} & \frac{2}{9} & \frac{2}{9} & -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & -\frac{1}{9} & -\frac{1}{9} & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} \end{bmatrix}$$

**9** **(a)** Let **G** be a generalized inverse of **A.** A generalized inverse of **PAQ** is $\mathbf{Q}^{-1}\mathbf{GP}^{-1}$.

Indeed, $\mathbf{PAQQ}^{-1}\mathbf{GP}^{-1}\mathbf{PAQ} = \mathbf{PAGAQ} = \mathbf{PAQ}$.

**(b)** The generalized inverse is **GA** because **GAGA** = **GA.**

**(c)** If **G** is a generalized inverse of **A** then (1/k)**G** is a generalized inverse of $k$**A.** We have that $k\mathbf{A}(1/k)\mathbf{G}k\mathbf{A} = \mathbf{AGA} = \mathbf{A}$.

**(d)** The generalized inverse is **ABA** because $(\mathbf{ABA})(\mathbf{ABA})(\mathbf{ABA}) = (\mathbf{ABA})^2(\mathbf{ABA}) = (\mathbf{ABA})(\mathbf{ABA}) = \mathbf{ABA}$.

**(e)** If **J** is $n \times n$ then $\frac{1}{n^2}\mathbf{J}$ is a generalized inverse of **J**.

$$\mathbf{J}\frac{1}{n^2}\mathbf{JJ} = \mathbf{J}$$

**10** **(a)** The identity and zero matrix and idempotent matrices. Also three by three matrices that satisfy the characteristic equation $\mathbf{A}^3 - \mathbf{A} = \mathbf{0}$.

**(b)** Orthogonal matrices $\mathbf{AA'A} = \mathbf{A}$ because $\mathbf{A'A} = \mathbf{I}$.

**(c)** The identity matrix, the zero matrix, and an idempotent matrix.

**(d)** No matrices.

**(e)** Non-singular matrices

**11** Searle's definition means that for equations $\mathbf{Ax} = \mathbf{y}$ for a vector $\mathbf{t}$, $\mathbf{t'A} = \mathbf{0}$ implies $\mathbf{t'y} = \mathbf{0}$. For (a) $\mathbf{t'0} = \mathbf{0}$ for any vector $\mathbf{t}$. For (b) if $\mathbf{t'X'X} = \mathbf{0}$, implies $\mathbf{t'U\Lambda^{1/2}SS'\Lambda^{1/2}U'} = \mathbf{0}$. Multiply this by $\mathbf{U\Lambda^{-1/2}S}$ to get $\mathbf{t'U\Lambda^{1/2}S} = \mathbf{t'X'} = \mathbf{0}$.

**12** By substitution, we have

$$
\begin{aligned}
\tilde{\mathbf{x}} &= \mathbf{Gy} + (\mathbf{GA} - \mathbf{I})((\mathbf{G} - \mathbf{F})\mathbf{y} + (\mathbf{I} - \mathbf{FA})\mathbf{w}) \\
&= [\mathbf{G} + (\mathbf{GA} - \mathbf{I})(\mathbf{G} - \mathbf{F})]\mathbf{Ax} + (\mathbf{GA} - \mathbf{I})(\mathbf{I} - \mathbf{FA})\mathbf{w} \\
&= [\mathbf{GA} + \mathbf{GAGA} - \mathbf{GA} - \mathbf{GAFA} + \mathbf{FA}]\mathbf{x} + (\mathbf{GA} - \mathbf{I} - \mathbf{GAFA} + \mathbf{FA})\mathbf{w} \\
&= [\mathbf{GA} + \mathbf{GA} - \mathbf{GA} - \mathbf{GA} + \mathbf{FA}]\mathbf{x} + (\mathbf{GA} - \mathbf{I} - \mathbf{GA} + \mathbf{FA})\mathbf{w} \\
&= \mathbf{FAx} + (\mathbf{FA} - \mathbf{I})\mathbf{w} \\
&= \mathbf{Fy} + (\mathbf{FA} - \mathbf{I})\mathbf{w}.
\end{aligned}
$$

**13** The matrix $(\mathbf{I} - \mathbf{GA})$ is idempotent so it is its own generalized inverse. The requested solution is

$$
\begin{aligned}
\mathbf{w} &= (\mathbf{I} - \mathbf{GA})(\mathbf{G} - \mathbf{F})\mathbf{y} + (\mathbf{I} - \mathbf{GA})(\mathbf{FA} - \mathbf{I})\mathbf{z} \\
&= (\mathbf{GA} - \mathbf{FA} - \mathbf{GAGA} + \mathbf{GAFA})\mathbf{x} + (\mathbf{FA} - \mathbf{GAFA} - \mathbf{I} + \mathbf{GA})\mathbf{z} \\
&= (\mathbf{GA} - \mathbf{FA} - \mathbf{GA} + \mathbf{GA})\mathbf{x} + (\mathbf{FA} - \mathbf{GA} - \mathbf{I} + \mathbf{GA})\mathbf{z} \\
&= (\mathbf{G} - \mathbf{F})\mathbf{Ax} + (\mathbf{FA} - \mathbf{I})\mathbf{z} \\
&= (\mathbf{G} - \mathbf{F})\mathbf{y} + (\mathbf{FA} - \mathbf{I})\mathbf{z}.
\end{aligned}
$$

**14** **(a)** Since $\mathbf{A}$ has full-column rank, so does $\mathbf{A'A}$(see, for example, Gruber (2014) Theorem 6.4). Also $\mathbf{A'A}$ has full-row rank so it is non-singular. As a result, since $\mathbf{AGA} = \mathbf{A}$, $\mathbf{A'AGA} = \mathbf{A'A}$ and $\mathbf{GA} = \mathbf{I}$. Then $\mathbf{GAG} = \mathbf{G}$ and $\mathbf{GA}$ is a symmetric matrix.

**(b)** Since $\mathbf{A}$ has full-row rank, $\mathbf{A'}$ has full-column rank. Then $\mathbf{G'}$ is a left inverse of $\mathbf{A'}$ and $\mathbf{G'A'} = \mathbf{I}$, so $\mathbf{AG} = \mathbf{I}$ and $\mathbf{G}$ is a right inverse. Then $\mathbf{GAG} = \mathbf{G}$ and $\mathbf{AG}$ is a symmetric matrix.

**15** Suppose that the singular value decomposition of $\mathbf{A} = \mathbf{S'\Lambda^{1/2}U'}$. Then $\mathbf{A'A} = \mathbf{U\Lambda U'}$, $(\mathbf{A'A})^{\mathrm{p}} = (\mathbf{U\Lambda U'})(\mathbf{U\Lambda U'})\mathrm{L}(\mathbf{U\Lambda U'}) = \mathbf{U\Lambda^{p}U'}$. Then since $\mathbf{T}(\mathbf{A'A})^{r+1} = (\mathbf{A'A})^{r}$, $\mathbf{TU\Lambda^{r+1}U'} = \mathbf{U\Lambda^{r}U'}$. Post-multiply both sides of this equation by $\mathbf{U\Lambda^{-r}U'}$ to obtain $\mathbf{TU\Lambda U'} = \mathbf{UU'}$. Now post-multiply both sides by $\mathbf{U\Lambda^{-1/2}S}$ so that $\mathbf{TU\Lambda U'U\Lambda^{1/2}S} = \mathbf{UU'U\Lambda^{1/2}S} = \mathbf{U\Lambda^{1/2}S}$ and thus $\mathbf{TA'AA'} = \mathbf{A'}$.

**16** Any singular idempotent matrix would have the identity matrix for a generalized inverse. For example, $\mathbf{M} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

**17** Assume that $\mathbf{B}^-\mathbf{A}^-$ is a generalized inverse of $\mathbf{AB}$. Then

$$\mathbf{ABB}^-\mathbf{A}^-\mathbf{AB} = \mathbf{AB}$$

Pre-multiply the above equation by $\mathbf{A}^-$ and post-multiply it by $\mathbf{B}^-$. Then

$$\mathbf{A}^-\mathbf{ABB}^-\mathbf{A}^-\mathbf{ABB}^- = \mathbf{A}^-\mathbf{ABB}^-$$

so that $\mathbf{A}^-\mathbf{ABB}^-$ is idempotent.

Now suppose that $\mathbf{A}^-\mathbf{ABB}^-$ is idempotent. Then

$$\mathbf{A}^-\mathbf{ABB}^-\mathbf{A}^-\mathbf{ABB}^- = \mathbf{A}^-\mathbf{ABB}^-$$

Pre-multiply this equation by $\mathbf{A}$ and post-multiply it by $\mathbf{B}$ to obtain

$$\mathbf{AA}^-\mathbf{ABB}^-\mathbf{A}^-\mathbf{ABB}^-\mathbf{B} = \mathbf{AA}^-\mathbf{ABB}^-\mathbf{B}.$$

By virtue of $\mathbf{AA}^-\mathbf{A} = \mathbf{A}$ and $\mathbf{BB}^-\mathbf{B} = \mathbf{B}$, we get

$$\mathbf{ABB}^-\mathbf{A}^-\mathbf{AB} = \mathbf{AB}$$

so that $\mathbf{B}^-\mathbf{A}^-$ is a generalized inverse of AB.

**18** See Exercise 15 for an example where a matrix and its generalized inverse are not of the same rank.

First assume that $\mathbf{G}$ is a reflexive generalized inverse of $\mathbf{A}$. From $\mathbf{AGA} = \mathbf{A}$ rank($\mathbf{A}$) $\leq$ rank ($\mathbf{G}$). Likewise from $\mathbf{GAG} = \mathbf{G}$ rank ($\mathbf{G}$) $\leq$ rank ($\mathbf{A}$) so that rank($\mathbf{G}$) = rank($\mathbf{A}$).

On the other hand suppose that $\mathbf{G}$ is a generalized inverse of $\mathbf{A}$ with the same rank $r$ as $\mathbf{A}$. We can find non-singular matrices $\mathbf{P}$ and $\mathbf{Q}$ where

$$\mathbf{PAQ} = \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \text{ and as a result, } \mathbf{A} = \mathbf{P}^{-1} \begin{bmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{Q}^{-1}$$

A generalized inverse takes the form

$$\mathbf{G} = \mathbf{Q} \begin{bmatrix} \mathbf{I}_r & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix} \mathbf{P}$$

Because $\mathbf{G}$ has rank $r$ and the first $r$ columns are linearly independent, $\mathbf{C}_{22} = \mathbf{C}_{12}\mathbf{C}_{21}$. The verification that $\mathbf{G}$ is a reflexive generalized inverse follows by straightforward matrix multiplication.

**19** We have that

$$\begin{aligned}\mathbf{AGA} &= \mathbf{P}^{-1}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}\mathbf{Q}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix}\mathbf{P}\mathbf{P}^{-1}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}\\ &= \mathbf{P}^{-1}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}\\ &= \mathbf{P}^{-1}\begin{bmatrix}\mathbf{I} & \mathbf{DX}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1} = \mathbf{P}^{-1}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1} = \mathbf{A}.\end{aligned}$$

Thus, **A** is a generalized inverse of **G**. Also

$$\begin{aligned}\mathbf{GAG} &= \mathbf{Q}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix}\mathbf{P}\mathbf{P}^{-1}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{Q}^{-1}\mathbf{Q}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix}\mathbf{P}\\ &= \mathbf{Q}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix}\mathbf{P}\\ &= \mathbf{Q}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{YD} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{Z}\end{bmatrix}\mathbf{P}\\ &= \mathbf{Q}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{X}\\ \mathbf{Y} & \mathbf{YDX}\end{bmatrix}\mathbf{P} = \mathbf{G}\end{aligned}$$

if $\mathbf{YDX} = \mathbf{Z}$.

In Exercise 1, a generalized inverse for **B** could be

$$\mathbf{G} = \mathbf{Q}\begin{bmatrix}1 & 0 & 0 & 1\\ 0 & -\frac{1}{3} & 0 & 2\\ 0 & 0 & \frac{1}{2} & 3\\ 1 & 2 & 3 & 4\end{bmatrix}\mathbf{P} = \begin{bmatrix}-\frac{91}{6} & -\frac{25}{3} & \frac{31}{2} & -32\\ \frac{31}{3} & \frac{17}{3} & -13 & 32\\ 0 & 0 & 2 & -8\\ \frac{7}{2} & 2 & -\frac{7}{2} & 7\end{bmatrix}$$

This matrix is non-singular. However, **B** is a $4 \times 4$ matrix of rank 3.

**20** **(a)** A generalized inverse of **AB** would be $\mathbf{B}'\mathbf{G}$. Notice that

$$\mathbf{ABB}'\mathbf{GAB} = \mathbf{AIGAB} = \mathbf{AGABB} = \mathbf{AB}.$$

**(b)** Let **G** be a generalized inverse of **L**. Then a generalized inverse of **L**A would be $\mathbf{A}^{-1}\mathbf{G}$. Observe that

$$\mathbf{LAA}^{-1}\mathbf{GLA} = \mathbf{LGLA} = \mathbf{LA}.$$

**21** The matrix itself.

**22** **(a)** Let $\mathbf{H}$ be a generalized inverse different from $\mathbf{G}$. We must find an $\mathbf{Z}$ so that

$$\mathbf{H} = \mathbf{G} + \mathbf{Z} - \mathbf{GAZAG}. \text{ Let } \mathbf{Z} = \mathbf{H} - \mathbf{G} + \mathbf{GAG}. \text{ Then}$$
$$\mathbf{G} + \mathbf{H} - \mathbf{G} + \mathbf{GAG} - \mathbf{GA}(\mathbf{H} - \mathbf{G} + \mathbf{GAG})\mathbf{AG}$$
$$= \mathbf{H} + \mathbf{GAG} - \mathbf{GAHAG} + \mathbf{GAGAG} - \mathbf{GAGAGAG}$$
$$= \mathbf{H} + \mathbf{GAG} - \mathbf{GAG} + \mathbf{GAG} - \mathbf{GAG} = \mathbf{H}.$$

**(b)** If we can generate all generalized inverses, we generate all solutions.

**23** **(a)** We have that

$$\begin{bmatrix}\mathbf{U} & \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathbf{C}_1\\ \mathbf{C}_2 & \mathbf{C}_3\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix}\begin{bmatrix}\mathbf{S}' & \mathbf{T}'\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{1/2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{U}'\\ \mathbf{V}'\end{bmatrix}\begin{bmatrix}\mathbf{U} & \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathbf{C}_1\\ \mathbf{C}_2 & \mathbf{C}_3\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{U} & \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathbf{C}_1\\ \mathbf{C}_2 & \mathbf{C}_3\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{1/2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathbf{C}_1\\ \mathbf{C}_2 & \mathbf{C}_3\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{U} & \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{I} & 0\\ \mathbf{C}_2\mathbf{\Lambda}^{1/2} & 0\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathbf{C}_1\\ \mathbf{C}_2 & \mathbf{C}_3\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix} = \begin{bmatrix}\mathbf{U} & \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathbf{C}_1\\ \mathbf{C}_2 & \mathbf{C}_2\mathbf{\Lambda}^{1/2}\mathbf{C}_1\end{bmatrix}$$

if and only if $\mathbf{C}_3 = \mathbf{C}_2\mathbf{\Lambda}^{1/2}\mathbf{C}_1$.

**(b)** Observe that

$$\mathbf{X} = \begin{bmatrix}\mathrm{S}' & \mathrm{T}'\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{1/2} & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathrm{U}'\\ \mathrm{V}'\end{bmatrix} \text{ and } \mathbf{G} = \begin{bmatrix}\mathrm{U} & \mathrm{V}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathrm{C}_1\\ \mathrm{C}_2 & \mathrm{C}_3\end{bmatrix}\begin{bmatrix}\mathrm{S}\\ \mathrm{T}\end{bmatrix}.$$

Then

$$\mathbf{GX} = \begin{bmatrix}\mathbf{U} & \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathbf{C}_1\\ \mathbf{C}_2 & \mathbf{C}_3\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix}\begin{bmatrix}\mathbf{S}' & \mathbf{T}'\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{1/2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{U}'\\ \mathbf{V}'\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{U}\ \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{C}_2\mathbf{\Lambda}^{1/2} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{U}'\\ \mathbf{V}'\end{bmatrix}$$

is symmetric if and only if $\mathbf{C}_2 = 0$.

**(c)** Observe that

$$\mathbf{XG} = \begin{bmatrix}\mathbf{S}' & \mathbf{T}'\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{1/2} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{U}'\\ \mathbf{V}'\end{bmatrix}\begin{bmatrix}\mathbf{U} & \mathbf{V}\end{bmatrix}\begin{bmatrix}\mathbf{\Lambda}^{-1/2} & \mathbf{C}_1\\ \mathbf{C}_2 & \mathbf{C}_3\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix}$$
$$= \begin{bmatrix}\mathbf{S}' & \mathbf{T}'\end{bmatrix}\begin{bmatrix}\mathbf{I} & \mathbf{\Lambda}^{1/2}\mathbf{C}_1\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{S}\\ \mathbf{T}\end{bmatrix}$$

is symmetric if and only if $\mathbf{C}_1 = 0$.

**24** For $\mathbf{M}$, we have

$$\begin{aligned}
\mathbf{XMX} &= \mathbf{X}(\mathbf{X}'\mathbf{X})^{+}\mathbf{X}'\mathbf{X} = \mathbf{X} \text{ by Theorem 10}\\
\mathbf{MXM} &= (\mathbf{X}'\mathbf{X})^{+}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{+}\mathbf{X}' = (\mathbf{X}'\mathbf{X})^{+}\mathbf{X}'\\
\mathbf{XM} &= \mathbf{X}(\mathbf{X}'\mathbf{X})^{+}\mathbf{X}', \text{ a symmetric matrix by Theorem 10}\\
\mathbf{MX} &= (\mathbf{X}'\mathbf{X})^{+}\mathbf{X}'\mathbf{X}, \text{ a symmetric matrix by Penrose axiom applied to } \mathbf{X}'\mathbf{X}
\end{aligned}$$

Using the singular value decomposition recall that if $\mathbf{X} = \mathbf{S}'\mathbf{\Lambda}^{1/2}\mathbf{U}', \mathbf{X}^{+} = \mathbf{U}\mathbf{\Lambda}^{-1/2}\mathbf{S}$. Then

$$\mathbf{M} = (\mathbf{X}'\mathbf{X})^{+}\mathbf{X}' = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}'\mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{S} = \mathbf{U}\mathbf{\Lambda}^{-1/2}\mathbf{S}.$$

For $\mathbf{W}$, we have

$$\begin{aligned}
\mathbf{XWX} &= \mathbf{XX}'(\mathbf{XX}')^{+}\mathbf{X} = \mathbf{X} \text{ applying Theorem 10 to } \mathbf{X}',\\
\mathbf{WXW} &= \mathbf{X}'(\mathbf{XX}')^{+}\mathbf{XX}'(\mathbf{XX}')^{+} = \mathbf{X}'(\mathbf{XX}')^{+}, \text{ by the reflexivity of } (\mathbf{XX}')^{+},\\
\mathbf{XW} &= \mathbf{XX}'(\mathbf{XX}')^{+} \text{ applying the Penrose condition to } \mathbf{XX}',\\
\mathbf{WX} &= \mathbf{X}'(\mathbf{XX}')^{+}\mathbf{X} \text{ by Theorem 10.}
\end{aligned}$$

Using the singular value decomposition $\mathbf{W} = \mathbf{X}'(\mathbf{XX}')^{+} = \mathbf{U}\mathbf{\Lambda}^{1/2}\mathbf{SS}'\mathbf{\Lambda}^{-1}\mathbf{S} = \mathbf{U}\mathbf{\Lambda}^{-1/2}\mathbf{S} = \mathbf{X}^{+}$.

**25** By direct verification of Penrose conditions

$$\begin{aligned}
&\mathbf{UNU}'\mathbf{UN}^{-1}\mathbf{U}'\mathbf{UNU}' = \mathbf{UNN}^{-1}\mathbf{NU}' = \mathbf{UNU}',\\
&\mathbf{UN}^{-1}\mathbf{U}'\mathbf{UNU}'\mathbf{UN}^{-1}\mathbf{U}' = \mathbf{UN}^{-1}\mathbf{NN}^{-1}\mathbf{U}' = \mathbf{UN}^{-1}\mathbf{U}',\\
&\mathbf{UN}^{-1}\mathbf{U}'\mathbf{UNU}' = \mathbf{UU}', \text{ a symmetric matrix, and}\\
&\mathbf{UNU}'\mathbf{UN}^{-1}\mathbf{U}' = \mathbf{UU}', \text{ a symmetric matrix.}
\end{aligned}$$

**26** Again by direct verification of the Penrose axioms

$$\begin{aligned}
&\mathbf{PAP}'\mathbf{PA}^{+}\mathbf{P}'\mathbf{PAP}' = \mathbf{PAA}^{+}\mathbf{AP}' = \mathbf{PAP}',\\
&\mathbf{PA}^{+}\mathbf{P}'\mathbf{PAP}'\mathbf{PA}^{+}\mathbf{P}' = \mathbf{PA}^{+}\mathbf{AA}^{+}\mathbf{P}' = \mathbf{PA}^{+}\mathbf{P}',\\
&\mathbf{PAP}'\mathbf{PA}^{+}\mathbf{P}(\mathbf{PA}^{+}\mathbf{P}'\mathbf{PAP}')' = (\mathbf{PA}^{+}\mathbf{AP}')' = \mathbf{P}(\mathbf{A}^{+}\mathbf{A})'\mathbf{P}' = \mathbf{PA}^{+}\mathbf{AP}'
\end{aligned}$$

and similarly

$$(\mathbf{PAP}'\mathbf{PA}^{+}\mathbf{P}')' = \mathbf{P}(\mathbf{AA}^{+})'\mathbf{P}' = \mathbf{PAA}^{+}\mathbf{P}'.$$

**27** **(a)** Using the singular value decomposition of $\mathbf{X}$

$$\mathbf{X}^{+}(\mathbf{X}^{+})' = \mathbf{U}\mathbf{\Lambda}^{-1/2}\mathbf{S}(\mathbf{U}\mathbf{\Lambda}^{-1/2}\mathbf{S})' = \mathbf{U}\mathbf{\Lambda}^{-1/2}\mathbf{SS}'\mathbf{\Lambda}^{-1/2}\mathbf{U}' = \mathbf{U}\mathbf{\Lambda}^{-1}\mathbf{U}' = (\mathbf{X}'\mathbf{X})^{+}.$$

**(b)** Again using the singular value decomposition of $\mathbf{X}$

$$(\mathbf{X}')^{+}\mathbf{X}^{+} = \mathbf{S}'\mathbf{\Lambda}^{-1/2}\mathbf{U}'\mathbf{U}\mathbf{\Lambda}^{-1/2}\mathbf{S} = \mathbf{S}'\mathbf{\Lambda}^{-1}\mathbf{S} = (\mathbf{XX}')^{+}.$$